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Causality in non-commutative quantum field theories

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Abstract
We study causality in noncommutative quantum field theory with a space–space noncommutativity. We employ the S operator approach of Bogoliubov–Shirkov (BS). We generalize the BS criterion of causality to the noncommutative theory. The criterion to test causality leads to a nonzero difference between the $T*$ product and the T product as a condition of causality violation for a spacelike separation. We discuss two examples; one in a scalar theory and another in the Yukawa theory. In particular, in the context of a noncommutative Yukawa theory, with the interaction Lagrangian $\overline{\psi}(x) \star \psi(x) \star \phi(x)$, is observed to be causality violating even in the case of space–space noncommutativity for which $\theta^{0i} = 0$.

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1. Introduction

Nonlocal field theories, in a variety of forms, have been proposed from time to time as a possible remedy against the UV divergences that arise due to the ill-defined product of the fields at an identical spacetime point. Noncommutative space was first introduced, with a similar goal, by Snyder [1]. Later, noncommutative spaces were found to arise in several different contexts. Interplay between quantum theory and gravitation suggests a non-trivial structure of spacetime at short distances and a noncommutative structure of spacetime is a possibility. Indeed, the notion of spacetime as a c^{∞} manifold may not exist down to the distance scales of the order of Plank length scale [2]. Spacetime noncommutativity naturally appears as a low-energy limit of the open string theory on a D-brane configuration in a constant B-field background [3].

We shall deal with noncommutative field theories defined on a noncommutative manifold obeying

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ is a constant real antisymmetric matrix of length dimension two[1]. Given a local field theory on a commutative spacetime, it can be generalized to a noncommutative spacetime. In the net effect, it amounts to a replacement of ordinary local product by a Moyal star product of the two functions [4]

$$(A * B)(x) := \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y} A(x)B(y)|_{y=x}. \tag{2}$$

Noncommutative quantum field theory (NCQFT) is constructed out of the action with the usual product of the fields replaced by the Moyal star product. Nonlocality of the theory stems from the Moyal product which consists of a tower of an arbitrarily high number of spacetime derivatives. The nonlocality enters through only the interaction terms because at the quadratic level both the commutative and noncommutative theories are identical. This is reflected from the fact that Moyal star product of the two fields can always be decomposed into the usual product of the fields and a total derivative term which would be zero if integrated over all spacetime. Noncommutative quantum field theory [4] is in effect, a *nonlocal* quantum field theory, as is especially obvious from its star-product formulation. A typical nonlocal QFT has its interaction spread over a finite region at a given instant and thus this includes points that are separated by a spacelike separation. This makes possible for a violation of causality in such theories. The violation of causality has been a subject of much discussion. Definitions of causality can be/have been attempted at various levels:

- The primarily meaningful definition of causality violation is linked with its experimental observation.
- Causality can also be tested in a physical situation by whether the physical cause precedes the physical effect [5].
- However, alternate definitions are often given in terms of the *micro-causality*: i.e. whether a given commutator (or a related object) of 'local' observables vanishes outside the light cone [6–8]. See also [9].
- There have been attempts to link causality with dispersion relation approach also for NCQFT. [10]

For a different perspective on causality, however, see e.g. [11].

In the context of a noncommutative quantum field theory there are no (strictly) local observables as elucidated below. Hence, the question of definition of micro-causality is a moot question and indeed various definitions of micro-causality itself have been proposed in the context of a NCQFT.

Consider first a *local* field theory. Suppose, we are given a local observable, $O[\phi(x)]$ in fields (generically denoted by ϕ), and their *finite*-order time derivatives. In a NCQFT, it will be represented by a star product of the fields and their derivatives. For example, $O_1[\phi] = \phi^3(x)$ will be represented by an observable $O_1^*[\phi] = \phi(x) \star \phi(x) \star \phi(x)$ and $O_2[\phi] = \phi\partial_\mu\phi + \partial_\mu\phi\phi$ will be represented by an observable $O_2^*[\phi] = \phi(x) \star \partial_\mu\phi(x) + \partial_\mu\phi(x) \star \phi(x)$. For the sake of brevity and convenience, we shall call these also as 'local' (in quotes) in the context of a NCQFT.

We first enumerate below the definitions of micro-causality suggested:

(1) For two 'local' observables $O_1^*(x)$ and $O_2^*(y)$, the theory violates micro-causality [6] if

$$[O_1^*(x), O_2^*(y)] \neq 0, \qquad \text{for} \quad (x-y)^2 < 0. \tag{3}$$

Here, $[O_1^*(x), O_2^*(y)]$ stands for the *commutator*. We shall use $x \sim y$ to imply that x and y are spacelike separated.

[1] We shall not necessarily commit ourselves to a value/scale for the parameter θ, but leave it as a free parameter to be determined experimentally.

(2) For two 'local' observables $O_1^*(x)$ and $O_2^*(y)$, the theory violates micro-causality if

$$[O_1^*(x), O_2^*(y)]_* \neq 0, \qquad \text{for} \quad (x-y)^2 < 0. \tag{4}$$

Here, $[O_1^*(x), O_2^*(y)]_*$ stands for the *star-commutator* [7] defined by

$$[O_1^*(x), O_2^*(y)]_* = O_1^*(x) \star O_2^*(y) - O_2^*(y) \star O_1^*(x) \tag{5}$$

with

$$O_1^*(x) \star O_2^*(y) \equiv \exp\left\{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right\} O_1^*(x) O_2^*(y)$$

for an *arbitrary* pair of points x, y.

In [9], on the other hand, it has been suggested, in the context of the space–space noncommutativity, that the micro-causality condition for elementary field operators be imposed outside the light-wedge and not the light-cone:

$$[\phi(x), \phi(y)] = 0 \qquad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0 \tag{6}$$

where light-wedge being defined in terms of the commuting coordinates: $(x_0 - y_0)^2 - (x_3 - y_3)^2 = 0$.

In this work, we shall attempt to look at the problem of causality from another perspective. This is based on the approach by Bogoliubov and Shirkov (BS) [12]. They have formulated a general S operator approach (however, for a commutative spacetime) that does not require one to commit to a specific field theory setting and is based upon a primary definition of causality: a physical disturbance cannot propagate out of its forward light-cone. This approach thus has a direct physical basis and has been found useful in nonlocal quantum field theories [13]. We generalize this approach, as far as it is possible, to a space–space NCQFT and develop a criterion based on this approach to test causality. In section 2, we first summarize works related to the causality. Generalization of the BS approach requires that we introduce a spacetime-dependent coupling in the intermediate stages. In section 3, we first carry out this generalization. Another issue that differs from the BS approach to the commutative field theory is that the use of spacelike intervals needs a reformulation. In this section, we go over the argument for BS criterion for a NCQFT to see where it needs a revision. In section 4, we arrive at a criterion to test causality. In section 5, we work out two examples, one in the scalar NCQFT and another in the Yukawa NCQFT for causality violation (CV). We show that in either cases, even for the space–space noncommutativity, there is CV. In section 6, we shall connect the causality criterion to the compatibility of measurement process for two 'local' observables.

2. Summary of earlier works

We shall briefly summarize earlier works, augmenting what has partly been said in section 1. Should micro-causality be valid for a theory, we expect that *any* pair of 'local' observables, $O_1^*(x)$ and $O_2^*(y)$, should commute for a spacelike separation

$$[O_1^*(x), O_2^*(y)] = 0 \qquad \text{for} \quad x \sim y. \tag{7}$$

In particular, we expect that (7) should hold as an operator equation, i.e. *every* matrix element of $[O_1^*(x), O_2^*(y)]$ should vanish in such a case.

Original works are based on the above definition of micro-causality. Chaichian *et al* [6] observed that the matrix element

$$\langle 0|[: \phi(\mathrm{x}) \star \phi(\mathrm{x}) :, : \phi(\mathrm{y}) \star \phi(\mathrm{y}) :]_{x_o=y_o}|p, p'\rangle$$

is nonzero for $\theta^{0i} \neq 0$ and thus violates micro-causality in the spacetime NCQFT; and generalized the idea to light-like noncommutativity: $\theta^{\mu\nu}\theta_{\mu\nu} = 0$.

On the other hand, Greenberg [7] has calculated the following matrix element of the following commutator:

$$\langle 0|[: \varphi(x) \star \varphi(x) :, \partial_0 (: \varphi(y) \star \varphi(y) :)]_{x_o=y_o}|p, p'\rangle$$

and drew attention to the fact that it fails to obey micro-causality even for the case $\theta^{0i} = 0$, i.e. for the space–space noncommutativity.

Greenberg then introduced the Moyal (star) commutator (see (5)) and analyzed the quantity $\langle 0|[: \phi(\mathrm{x}) \star \phi(\mathrm{x}) :, : \phi(\mathrm{y}) \star \phi(\mathrm{y}) :]^{*}_{x_0=y_0}|p, p'\rangle$ which violates micro-causality even in the case of space–space noncommutativity in which $\theta^{0i} = 0$. He noted that the star commutator, unlike the ordinary commutator, is sensitive to the separation of x and y through Moyal phases and suggested that it is the star commutator, and not the commutator, that is relevant for micro-causality.

Zheng [8] has studied the star commutator further. He has calculated the vacuum expectation value of equal-time star commutator $\langle 0|[: \phi(x) \star \phi(x) :, : \phi(y) \star \phi(y) :]^{*}_{x_0=y_0}|0\rangle$ which vanishes for $\theta^{0i} = 0$. Zheng also studied the vacuum and the non-vacuum matrix elements of the quantity $[: \bar{\psi}_\alpha(x) \star \psi_\beta(x) :, : \bar{\psi}_\sigma(y) \star \psi_\tau(y) :]^{*}_{x_0=y_0}$ and showed that it does not vanish for spacelike separation, no matter whether $\theta^{0i} = 0$ or $\theta^{0i} \neq 0$.

As pointed out in the introduction, [9] has suggested use of light-wedge for elementary field micro-causality (see (6)) and has explored this in the case of the interacting fields. It has been shown that micro-causality commutator vanishes outside the light-wedge, and not the light-cone.

3. Development of criterion for causality

We shall first develop a criterion for causality violation along the lines of [12] Bogoliubov–Shirkov, appropriately generalized to a noncommutative (NC) spacetime. This will also enable us to construct a quantity that will enable us to decide under what conditions are two observables compatible (as further discussed in section 6). We shall restrict ourselves to a noncommutative spacetime with $\theta^{12} = -\theta^{21} \equiv \theta \neq 0$ and $\theta^{\mu\nu} = 0$ otherwise. In this frame of reference, time t is well defined and this makes a generalization of the BS criterion easier for such NC quantum field theories.

The BS discussion begins with an S operator. For the formulation of the BS criterion, we need a variable coupling $g(x)$ that can be varied over the spacetime, and the S operator, $S[g]$, for such a coupling[2]. Before we proceed with the generalization for the case of NCQFT, we shall first generalize the interaction term to include a variable $g(x)$.

3.1. Interaction term

Let the interaction of a local commutative field theory be

$$S_I = g \int \mathrm{d}^4x\, \mathcal{L}_I[\phi(x)].$$

In the spacetime-dependent coupling formalism, it would be replaced by

$$S'_I = \int \mathrm{d}^4x\, g(x)\mathcal{L}_I[\phi(x)].$$

[2] The idea of a variable coupling, at least over time, is not new: it is employed in the LSZ formulation.

In a noncommutative spacetime, this would be replaced by

$$S_I \to \int \mathrm{d}^4x\, g(x) \star \mathcal{L}_I^*[\phi(x)].$$

Here, $\mathcal{L}_I^*[\phi(x)]$ is a short-hand notation for the interaction $\mathcal{L}_I[\phi(x)]$ converted to a noncommutative space star product. For example, $S'_I = \int \mathrm{d}^4x\, g(x)\phi^4(x)$ would be replaced by

$$\int \mathrm{d}^4x\, g(x)\phi^4(x) \to \int \mathrm{d}^4x\, g(x) \star \phi(x) \star \phi(x) \star \phi(x) \star \phi(x).$$

It is easy to verify however that,

$$\int \mathrm{d}^4x\, g(x) \star \phi(x) \star \phi(x) \star \phi(x) \star \phi(x) \equiv \int \mathrm{d}^4x\, g(x)\phi(x) \star \phi(x) \star \phi(x) \star \phi(x). \tag{8}$$

To see this, consider the expression on the left-hand side of (8), written in momentum space,

$$\begin{aligned}
&\int \mathrm{d}^4x\, g(x) \star \phi(x) \star \phi(x) \star \phi(x) \star \phi(x) \\
&\quad = \int \mathrm{d}^4k\, \mathrm{d}^4k_1\, \mathrm{d}^4k_2\, \mathrm{d}^4k_3\, \mathrm{d}^4k_4 \exp\left\{\frac{-\mathrm{i}\theta^{\mu\nu}[k_\mu\,(k_1+k_2+k_3+k_4)_\nu + \mathrm{O.T.}]}{2}\right\} \\
&\qquad \times g\,(k)\,\phi\,(k_1)\,\phi\,(k_2)\,\phi\,(k_3)\,\phi\,(k_4)\,\delta^4\,(k+k_1+k_2+k_3+k_4) \\
&\quad = \int \mathrm{d}^4k\, \mathrm{d}^4k_1\, \mathrm{d}^4k_2\, \mathrm{d}^4k_3\, \mathrm{d}^4k_4 g\,(k) \exp\left\{\frac{-\mathrm{i}\theta^{\mu\nu}\,[\mathrm{O.T.}]}{2}\right\} \\
&\qquad \times \phi\,(k_1)\,\phi\,(k_2)\,\phi\,(k_3)\,\phi\,(k_4)\,\delta^4\,(k+k_1+k_2+k_3+k_4) \\
&\quad = \int \mathrm{d}^4x\, g(x)\phi(x) \star \phi(x) \star \phi(x) \star \phi(x).
\end{aligned}$$

In the second line, O.T. stands for other terms in the exponent not containing k and in third line, we have used $\theta^{\mu\nu}k_\mu k_\nu \equiv 0$. Thus, for example, $\frac{\delta S_I}{\delta g(x)} = \phi(x) \star \phi(x) \star \phi(x) \star \phi(x)$. We note that the S matrix in the lowest non-trivial order is gS_1 and is entirely generated by the tree-order matrix elements of S_I. Unitarity of the S matrix to this order implies,

$$(1+gS_1)^\dagger(1+gS_1) = 1 + O(g^2)$$

which leads to

$$S_1^\dagger = -S_1 = -\mathrm{i}S_I.$$

3.2. Spacelike intervals

In the discussion of the BS criterion of causality for a commutative spacetime, use is often made of spacelike intervals: $(x-x')^2 < 0$. On the noncommutative spaces in question, the $x-y$ coordinates do not commute. As such, we need to consider $(x-x')^2$ as an operator. We can still consider two spacetime points which are specified by *definite* values both for x_0, x_3 and x'_0, x'_3. However, $X \equiv x - x'$ and $Y \equiv y - y'$ are both operators. It is not difficult to see that

$$\begin{aligned}
X^2 + Y^2 &= (X+\mathrm{i}Y)(X-\mathrm{i}Y) + \mathrm{i}[X,Y] \\
&= (X-\mathrm{i}Y)(X+\mathrm{i}Y) - \mathrm{i}[X,Y] \\
\langle X^2 + Y^2\rangle &\geqslant |[X,Y]| \\
&= 2\theta
\end{aligned} \tag{9}$$

in view of the positive semi-definiteness of the operators $(X+\mathrm{i}Y)(X-\mathrm{i}Y)$ and $(X-\mathrm{i}Y)(X+\mathrm{i}Y)$. Thus, we shall regard an interval $(x-x')$ spacelike if $(x_0-x'_0)^2 - (x_3-x'_3)^2 < 2\theta$.

- For a spacelike separation with $(x_0 - x_0')^2 - (x_3 - x_3')^2 < 0$, it is possible to change the order of time coordinates x_0 and x_0' by a Lorentz transformation confined to the $z - t$ plane alone. Such Lorentz transformations preserve the nature of noncommutativity (i.e. space–space). We shall call such a spacelike separation a 'restricted' one and shall denote it by $x \asymp y$.
- Two distinct events x, x' with $x_0 = x_0'$ and $x_3 = x_3'$ can always be enclosed in *some* disjoint neighborhoods in this plane, compatible with $\Delta x_1 \Delta x_2 = \frac{1}{2}\theta$. Hence, a theory cannot be causal if it necessarily allows instantaneous propagation of a signal from one to another.

  Consider two distinct events, x and x', with $x_0 = x_0'$ and $x_3 = x_3'$. If there the theory allows for a propagation of signal from x to x', the theory also allows propagation from x' to x and thus the identification of cause and effect itself becomes ambiguous. This would contradict the very notion of causality.

We shall see, in section 4, that the criterion of causality demands that the commutator of the interaction Lagrangian vanish over just the two sets of points we have discussed.

3.3. Generalization of BS criterion to a NCQFT

Let $\{|\alpha, \text{in}\rangle\}$ denote a complete set of scattering in-states. We shall consider a particular matrix element

$$S_{\beta\alpha} = \langle \beta, \text{in}|S[g]|\alpha, \text{in}\rangle.$$

For a constant g, $S_{\beta\alpha}$ has the perturbative expansion:

$$S_{\beta\alpha} = \delta_{\beta\alpha} + g S^{(1)}_{\beta\alpha} + \frac{g^2}{2!} S^{(2)}_{\beta\alpha} + \cdots$$

and for a variable $g(x)$, it has an expansion:

$$S_{\beta\alpha}[g] = \delta_{\beta\alpha} + \int \mathrm{d}^4x\, g(x) S^{(1)}_{\beta\alpha}(x) + \frac{1}{2!}\int \mathrm{d}^4x\, \mathrm{d}^4y\, g(x) g(y) S^{(2)}_{\beta\alpha}(x, y) + \cdots. \tag{10}$$

We want to generalize this to the noncommutative quantum field theories. We expect the second term on the right-hand side to be replaced by[3] (see also subsection 3.1)

$$\int \mathrm{d}^4x\, g(x) S^{(1)}_{\beta\alpha}(x) \to \mathrm{Tr}\big[g(\widehat{x})\, S^{(1)}_{\beta\alpha}(\hat{x})\big] = \int \mathrm{d}^4x\, g(x) \star S^{(1)*}_{\beta\alpha}(x)$$
$$\equiv \int \mathrm{d}^4x\, g(x) S^{(1)*}_{\beta\alpha}(x).$$

If we had a constant coupling, we would have replaced

$$\int \mathrm{d}^4x\, g S^{(1)}_{\beta\alpha}(x) \to \int \mathrm{d}^4x\, g S^{(1)*}_{\beta\alpha}(x),$$

i.e. the replacements in the two cases are identical: $S^{(1)}_{\beta\alpha}(x) \to S^{(1)*}_{\beta\alpha}(x)$. In a similar manner, the third term on the right-hand side is as follows:

$$\frac{1}{2!}\int \mathrm{d}^4x\, \mathrm{d}^4y\, g(x) g(y) S^{(2)}_{\beta\alpha}(x, y)$$
$$\to \frac{1}{2!}\int \mathrm{d}^4x\, \mathrm{d}^4y\, g(x) \star S^{(2)*}_{\beta\alpha}(x, y) \star g(y)$$
$$\equiv \frac{1}{2!}\int \mathrm{d}^4x \left\{ \int \mathrm{d}^4y\, g(x) * S^{(2)*}_{\beta\alpha}(x, y) \exp\left\{\frac{\mathrm{i}\theta^{\mu\nu}\partial^y_\mu \partial^{y_1}_\nu}{2}\right\} g(y_1)\bigg|_{y=y_1} \right\}.$$

[3] We recall that in a QFT, $S^{(1)}$ is a field operator.

We can now carry out the integration over y for a fixed x and find that the noncommutative phase cancels out. In a similar manner one can deal with the x integration and find that

$$\begin{aligned}
&\frac{1}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\, g(x)g(y)S^{(2)}_{\beta\alpha}(x,y)\\
&\qquad\to \frac{1}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\, g(x)\star S^{(2)*}_{\beta\alpha}(x,y)\star g(y)\\
&\qquad\equiv \frac{1}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\, g(x)g(y)S^{(2)*}_{\beta\alpha}(x,y).
\end{aligned}$$

This can be generalized to the remaining terms in (10).

Thus, in the noncommutative theory also, we have an expansion of the same form as the commutative case:

$$S_{\beta\alpha}[g] = \delta_{\beta\alpha} + \int \mathrm{d}^4x\, g(x)S^{(1)}_{\beta\alpha}(x) + \frac{1}{2!}\int \mathrm{d}^4x\,\mathrm{d}^4y\, g(x)g(y)S^{(2)}_{\beta\alpha}(x,y) + \cdots \tag{11}$$

where we have now dropped the star on $S^{(n)}_{\beta\alpha}$ as we shall employ (11) only for the NCQFT[4]. We shall employ henceforth.

We note in passing that it is not *necessary* to employ the S operator ($U(-\infty,\infty)$) in this formulation. This observation becomes relevant especially for a theory for which some of the S matrix elements may not exist because of infrared divergences. The formulation can alternately be given also in terms of the unitary time-evolution operator $U[-T, T'; g]$.

Let us now recall that we are considering a theory on a space with $\theta^{0i} = 0$ and that the time coordinate is well defined and we can order the spacetime points by their time coordinate.

The derivation of the BS condition of causality proceeds much the same way as for the commutative spacetime.

We define the coupling constant functions:

$$\begin{aligned}
g(x) &= g_2(x) \qquad T' > x_0 > 0\\
&= g_1(x) \qquad 0 > x_0 > -T;\\
G_2(x) &= g_2(x) \qquad T' > x_0 > 0\\
&= 0, \qquad \text{otherwise};\\
G_1(x) &= 0 \qquad T' > x_0 > 0\\
&= g_1(x) \qquad 0 > x_0 > -T.
\end{aligned}$$

Now, causality demands that the evolution for $0 > x_0 > -T$ is unaffected by the value of the coupling for $T' > x_0 > 0$. This fact is not contradicted by the $x - y$ noncommutativity. We recall that the matrix elements of U depend only on the *coupling constant function* $g(x)$ and not on $g(\hat{x})$. Thus, should causality hold,

$$U(-T, 0; g) = U(-T, 0; g_1) = U(-T, 0; G_1).$$

Also, $U(0, T'; G_1) \equiv \mathcal{I}$. Also, the *evolution* operator for $t > 0$ depends only on the coupling for $t > 0$. Hence,

$$U(0, T'; g) = U(0, T'; G_2).$$

Thus,

$$\begin{aligned}
U(-T, T'; g) &= U(0, T'; g)U(-T, 0; g)\\
&= U(0, T'; G_2)U(-T, 0; G_1)\\
&= U(-T, T; G_2)U(-T, T; G_1).
\end{aligned}$$

[4] The purpose of the star on $S^{(n)}_{\beta\alpha}$ was to remind us that the S operator is different for the local and the NCQFT.

In a similar manner, for

$$\begin{aligned} g'(x) &= g_2'(x) \qquad T' > x_0 > 0 \\ &= g_1(x) \qquad 0 > x_0 > -T; \end{aligned}$$

we have

$$U(-T, T'; g') = U(-T, T'; G_2')U(-T, T'; G_1)$$

where we have defined, in an analogous manner,

$$\begin{aligned} G_2'(x) &= g_2'(x) \qquad T' > x_0 > 0 \\ &= 0, \qquad \text{otherwise}. \end{aligned}$$

Then,

$$U(-T, T'; g')U^{\dagger}(-T, T'; g) = U(-T, T'; G_2')U^{\dagger}(-T, T'; G_2) \tag{12}$$

and is independent of values of $g_1(x)$ for $-T < x_0 < 0$. This is the BS condition of causality. We may write the above equation in the form

$$U(-T, T'; g(y) + \delta g(y))U^{\dagger}(-T, T'; g(y)) = 1 + \delta U(-T, T'; g(y))U^{\dagger}(-T, T'; g(y)), \tag{13}$$

where $\delta g(y) \neq 0$ for some $T' > y_0 > 0$. This expression need not depend upon the behavior of $g(x)$ for $-T < x_0 < 0$. So, we have

$$\frac{\delta}{\delta g(x)}\left(\frac{\delta U(g)}{\delta g(y)}U^{\dagger}(g)\right) = 0 \qquad \text{for} \quad x < y, \tag{14}$$

($x < y$ stands for $x_0 < y_0$). This is the expression of causality in terms of the unitary time-evolution operator.

In the case of commutative QFT, the above condition also holds for $x \sim y$; since in such a case, it is possible to make a Lorentz transformation to a frame in which $x_0 < y_0$ holds. In the present case, there is a restriction on the possible Lorentz transformation that preserves the nature of noncommutativity. From the discussion of subsection 3.2, it follows that equation (14) holds also for a 'restricted' spacelike separation $x \asymp x'$, i.e. with $(x_0 - x_0')^2 - (x_3 - x_3')^2 < 0$.

Further, for two distinct points x, y with $x_0 = y_0 = 0$ and $x_3 = y_3$, we note that the quantity $U(-T, T'; g(y) + \delta g(y))U^{\dagger}(-T, T'; g(y))$, for $\delta g \neq 0$ only at y, is not dependent on the value of g at such x. This follows from our observation in section 3.2 that such points cannot be connected by a signal if causality is always to be ensured. This leads to the validity of (14) also for such a pair of points.

We can express the matrix $\hat{U}$ in the form of functionals in powers of $g(x)$:

$$\begin{aligned} U[g] &= 1 + \sum_{n\geqslant 1}\frac{1}{n!}\int U_n(x_1, \ldots, x_n)g(x_1)\cdots g(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n, \\ &= 1 + \int U_1(x_1)g(x_1)\mathrm{d}x_1 + \int U_2(x_1, x_2)g(x_1)g(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 + \cdots, \end{aligned} \tag{15}$$

where $U_n(x_1, \ldots, x_n)$ is a symmetric operator with respect to all arguments, and depends upon the field operators and on their partial derivatives at the points $x_1, \ldots, x_n$.

Unitarity of $\hat{U}$ matrix, i.e. $U^{\dagger}[g]U[g] = 1$ leads to the condition, for each n, given by

$$\begin{aligned} &U_n(x_1, \ldots, x_n) + U_n^{\dagger}(x_1, \ldots, x_n) \\ &\quad + \sum_{1\leqslant k\leqslant n-1} P\left(\frac{x_1, \ldots, x_k}{x_{k+1}, \ldots, x_n}\right)U_k(x_1, \ldots, x_k)U_{n-k}^{\dagger}(x_{k+1}, \ldots, x_n) = 0. \end{aligned} \tag{16}$$

The symbol $P\left(\frac{x_1,\dots,x_k}{x_{k+1},\dots,x_n}\right)$ stands for the sum over the distinct ways of partitioning ($\frac{n!}{k!(n-k)!}$ in number) $\{x_1, x_2, x_3, \dots, x_n\}$ into two sets of k and $(n-k)$ (such as $\{x_1, x_2, x_3, \dots, x_k\}$ $\{x_{k+1}, \dots, x_n\}$).

Using (12), condition of causality can be expressed as

$$\begin{aligned} C_n(y, x_1, \dots, x_n) &= \mathrm{i}U_{n+1}(y, x_1, \dots, x_n) \\ &\quad + \mathrm{i} \sum_{0\leqslant k\leqslant n-1} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) U_{k+1}(y, x_1, \dots, x_k) U_{n-k}^{\dagger}(x_{k+1}, \dots, x_n) \\ &= 0. \end{aligned} \tag{17}$$

Now, causality condition for $n = 1, 2$ reads as

$$C_1(x, y) \equiv \mathrm{i}U_2(x, y) + \mathrm{i}U_1(x)U_1^{\dagger}(y) = 0 \tag{18}$$

$$C_2(x, y, z) \equiv \mathrm{i}U_3(x, y, z) + \mathrm{i}U_1(x)U_2^{\dagger}(y, z) + \mathrm{i}U_2(x, y)U_1^{\dagger}(z) + \mathrm{i}U_2(x, z)U_1^{\dagger}(y) = 0 \tag{19}$$

and unitary condition gives

$$U_1(x) + U_1^{\dagger}(x) = 0 \tag{20}$$

$$U_2(x, y) + U_2^{\dagger}(x, y) + U_1(x)U_1^{\dagger}(y) + U_1(y)U_1^{\dagger}(x) = 0. \tag{21}$$

The 'S' matrix can always be recovered from the unitary time-evolution operator in the large(infinite) time limit. So, we can have causality and unitarity condition for $n = 1,2$ like above as follows:

Causality condition

$$\mathrm{i}S_2(x, y) + \mathrm{i}S_1(x)S_1^{\dagger}(y) = 0 \tag{22}$$

$$\mathrm{i}S_3(x, y, z) + \mathrm{i}S_1(x)S_2^{\dagger}(y, z) + \mathrm{i}S_2(x, y)S_1^{\dagger}(z) + \mathrm{i}S_2(x, z)S_1^{\dagger}(y) = 0. \tag{23}$$

Unitarity condition

$$S_1(x) + S_1^{\dagger}(x) = 0 \tag{24}$$

$$S_2(x, y) + S_2^{\dagger}(x, y) + S_1(x)S_1^{\dagger}(y) + S_1(y)S_1^{\dagger}(x) = 0. \tag{25}$$

In particular, if the theory has T invariance, the S operator for the time-reversed theory is $S^{\dagger}$: $TST^{-1} = S^{\dagger}$. We now apply this to an analog of (15) for the S operator and invoke[5] $Tg(t, \mathbf{x})T^{-1} = g(-t, \mathbf{x})$. Then, we obtain,

$$S_2^{\dagger}(x, y) = \mathcal{I}_T S_2(-x_0, \mathbf{x}; -y_0, \mathbf{y}) \tag{26}$$

where $\mathcal{I}_T$ stands for the operation of changing the sign of time labels at the end of a calculation of a matrix element. Then, (25) implies,

$$S_2(x, y) + \mathcal{I}_T S_2(-x_0, \mathbf{x}; -y_0, \mathbf{y}) + S_1(x)S_1^{\dagger}(y) + S_1(y)S_1^{\dagger}(x) = 0. \tag{27}$$

We shall soon demonstrate that causality implies

$$S_2(0, \mathbf{x}; 0, \mathbf{y}) = \tfrac{1}{2}[S_1(x)S_1(y) + S_1(y)S_1(x)]\big|_{x_0=y_0=0} \tag{28}$$

which is compatible with (27) (note: $S_1^{\dagger} = -S_1$).

[5] If the original theory with a constant coupling has a time-reversal invariance, the intermediate action $S[g(x)]$, with a spacetime-dependent coupling $g(x)$, can be made time-reversal invariant, if we associate the following transformation for $g(x)$.

4. The BS causality criterion

As shown in section 3, the causality condition can be expressed along similar lines for a noncommutative quantum field theory as for the commutative one. One of these is

$$H_1\,(x,y) \equiv \mathrm{i}S_2\,(x,y) + \mathrm{i}S(x)S_1^\dagger(y) = \mathrm{i}S_2\,(x,y) - \mathrm{i}S_1(x)S_1(y) = 0, \qquad x >\asymp y. \tag{29}$$

This implies

$$S_2(x,y) = S_1(x)S_1(y) = -O^*(x)O^*(y) \qquad \text{for} \quad x >\asymp y \tag{30}$$

since, $S_1(x) = \mathrm{i}O^*(x)$. If we interchange x, y and use the symmetry of $S_2(x, y)$, we have

$$S_2(x,y) = S_1(y)S_1(x) = -O^*(y)O^*(x) \qquad \text{for} \quad y >\asymp x. \tag{31}$$

We now consider two points x, y such that $x \asymp y$. Then, for such a case, (30) and (31) lead to,

$$[S_1(x), S_1(y)] = 0, \qquad x \asymp y. \tag{32}$$

We shall now look at causality in a general case, i.e. we allow x, y to be arbitrary (we have left out the case of $x_0 = y_0$). From the remarks following (14), we note that (30) and (31) are valid also when $x_0 = y_0$ for $\mathbf{x} \neq \mathbf{y}$ (the case with $x_3 \neq y_3$ is already covered). Employing the symmetry of $S_2(x, y)$, we have

$$S_2(x,y) = \tfrac{1}{2}[S_1(x)S_1(y) + S_1(y)S_1(x)], \qquad x_0 = y_0, \quad \mathbf{x} \neq \mathbf{y}. \tag{33}$$

Combining (30), (31) and (33), we have a consequence of causality condition[6]

$$S_2(x,y) = \mathrm{i}^2 T[O^*(x)O^*(y)], \qquad \mathbf{x} \neq \mathbf{y}. \tag{34}$$

In addition, we also have,

$$[S_1(x_0,\mathbf{x}), S_1(y_0,\mathbf{y})] = 0, \qquad x_0 = y_0. \tag{35}$$

The above equation is valid at the set of points characterized by $x_0 = y_0$, $x_3 = y_3$ not included in the domain of validity of (32), namely $x \asymp y$. We observe that the criterion of causality requires that the interaction Lagrangian $\mathcal{L}_I(x)$ commute with itself, $\mathcal{L}_I(y)$, whenever $(x-y)^2$ fulfills either of the conditions mentioned in subsection 3.2. We also note that this consequence has *followed* from the primary meaning of causality employed in subsection 3.3 (together with other principles).

Now, equation (34), demanded by causality, may not always be obeyed. First we recall that when interaction term S_1 contains time derivatives, it is well known that time ordered product in (34) is not covariant. In QFT we often introduce another time ordered product, the T^* product which is covariant (in a commutative case). It is known that in the path integral formulation, we naturally generate Green's function of T^* ordered product of field operators. Assuming that the NCQFT is quantized using the path integral formulation, as is normally done, it will generate Green functions, covariant in appearance (if we were to look upon $\theta_{\mu\nu}$ as a tensor) and thus are not expected to coincide with those of (34). So, if we obtain a matrix element of

$$S_2(x,y) = \mathrm{i}^2 T^*[O^*(x)O^*(y)] \tag{36}$$

and find that it differs from that of $S_2(x, y)$ of equation (34) (dictated by causality), we can conclude that causality is violated. In other words,

$$\Delta \equiv T^*[O^*(x)O^*(y)] - T[O^*(x)O^*(y)] \tag{37}$$

can be used to test causality in a quantum field theory.

[6] We have adopted a symmetric definition for $\theta(x_0 - y_0)$: $\theta(x_0 - y_0) + \theta(y_0 - x_0) = 1 \Rightarrow \theta(0) = 1/2$.

We shall now elaborate on Δ of equation (37) and show that if we had only local interactions or higher-order derivative interaction terms of finite order, Δ is zero for $x \neq y$. We shall see that for a truly nonlocal field theory such as NCQFT or other nonlocal QFT, Δ may be nonzero. Let us consider two operators depending on $\phi(x)$ and its derivatives:

$$O_1(x) \equiv D_1[\varphi(x_1)\cdots\varphi(x_n)]|_{x_1=\cdots=x_n=x} \tag{38}$$

$$O_2(y) \equiv D_2[\varphi(y_1)\cdots\varphi(y_n)]|_{y_1=\cdots=y_n=y} \tag{39}$$

where D_1 and D_2 are as yet general operators that implement differentiation. Then,

$$\begin{aligned}
T^*[O_1(x)O_2(y)] \\
&= D_1D_2[\theta(X^0-Y^0)\varphi(x_1)\cdots\varphi(x_n)\varphi(y_1)\cdots\varphi(y_n) \\
&\quad +\theta(Y^0-X^0)\varphi(y_1)\cdots\varphi(y_n)\varphi(x_1)\cdots\varphi(x_n)]|_{x_1=\cdots=x_n=x,y_1=\cdots=y_n=y} \\
&= O_1(x)O_2(y) \\
&\quad + D_1D_2\{\theta(Y^0-X^0)[\varphi(y_1)\cdots\varphi(y_n),\varphi(x_1)\cdots\varphi(x_n)]\}|_{x_1=\cdots=x_n=x,y_1=\cdots=y_n=y} \\
&= T[O_1(x)O_2(y)]+\theta(y^0-x^0)[O_1(x),O_2(y)] \\
&\quad - D_1D_2\{\theta(Y^0-X^0)[\varphi(x_1)\cdots\varphi(x_n),\varphi(y_1)\cdots\varphi(y_n)]\}|_{x_1=\cdots=x_n=x,y_1=\cdots=y_n=y}.
\end{aligned}$$

With [14],

$$X^0 = \frac{\sum_{i=1}^n x_i^0}{n} \qquad \text{and} \qquad Y^0 = \frac{\sum_{i=1}^n y_i^0}{n}.$$

Thus,

$$\begin{aligned}
T^*[O_1(x)O_2(y)] - T[O_1(x)O_2(y)] \\
&= \theta(y^0-x^0)[O_1(x),O_2(y)] \\
&\quad - D_1D_2\{\theta(Y^0-X^0)[\varphi(x_1)\cdots\varphi(x_n),\varphi(y_1)\cdots\varphi(y_n)]\}|_{x_1=\cdots=x_n=x,y_1=\cdots=y_n=y}.
\end{aligned}$$

Suppose O_1, O_2 contain finite-order time derivatives. Then the above difference receives contributions only when one or more time derivatives in D_1 or D_2 act upon the theta function. This leads to a difference that contains $\delta(x^0-y^0)$ or finite-order derivative of $\delta(x^0-y^0)$. The commutators on the other hand lead to terms $\propto \delta^3(\mathbf{x}-\mathbf{y})$ or its finite-order derivatives. The terms therefore vanish whenever $x \neq y$. On the other hand, for a NCQFT, the operators D_1 and D_2 contain derivatives of an arbitrary order. For example, for $O_1 = O_2 = \phi\star\phi\star\cdots\star\phi$,

$$\begin{aligned}
D_1 &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}(\partial_\mu^{x_1}\partial_\nu^{x_2}+\cdots+\partial_\mu^{x_{n-1}}\partial_\nu^{x_n})} \\
D_2 &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}(\partial_\mu^{y_1}\partial_\nu^{y_2}+\cdots+\partial_\mu^{y_{n-1}}\partial_\nu^{y_n})}.
\end{aligned}$$

Such a series acting on $\delta^3(\mathbf{x}-\mathbf{y})$ smears it over a nonvanishing region in the $x_1 - x_2$ plane. This is illustrated in the following section.

5. Calculations for causality violation in NCQFT

We shall exhibit calculation of Δ, the difference between T^* product and T product, corresponding to the two different cases in the context of two different field theories:

- $O_1^*(x) = \frac{\dot{\varphi}(x)*\varphi(x)+\varphi(x)*\dot{\varphi}(x)}{2}$ in a scalar theory;
- $O_2^*(x) = \bar{\psi}(x)*\psi(x)*\varphi(x)$ in the Yukawa theory.

While O_1^* cannot be an interaction Lagrangian, being quadratic; this simple example will illustrate the more general case. The essential facet of both the operators is that they contain both a 'coordinate' and a 'momentum'. We shall be considering the space–space noncommutativity, i.e. $\theta^{0i} = 0$ throughout the calculations.

Example 1. Consider the case of the former operator $O_1^*(x)$. Now,

$$\begin{aligned}\langle 0|\Delta|pp'\rangle &= \langle 0|\{T^*[:O_1^*(x)::O_1^*(y):] - T[:O_1^*(x)::O_1^*(y):]\}|pp'\rangle \\ &= \langle 0|\tfrac{1}{4}\big\{\partial_0^x\delta(x^0-y^0)[:\varphi(y)*\varphi(y):,:\varphi(x)*\varphi(x):]. \\ &\quad + \delta(x^0-y^0)\big[:\varphi(y)*\varphi(y):,\partial_0^x(:\varphi(x)*\varphi(x):)\big] \\ &\quad - \delta(x^0-y^0)\big[\partial_0^y(:\varphi(y)*\varphi(y):),:\varphi(x)*\varphi(x):\big]\big\}|pp'\rangle \end{aligned} \tag{40}$$

The right-hand side of above equation (40) has three terms. The commutator in the first term can be Taylor expanded around $x_0 = y_0$. The leading term (with $x_0 = y_0$) is zero for $\theta^{0i} = 0$, as shown by Chaichian *et al* [6]; and in the second term, we use $(x_0-y_0)\partial_0^x\delta(x^0-y^0) = -\delta(x^0-y^0)$. It then cancels second term. Possible nonzero contribution comes from the third term. Consider the case for which $\theta^{12} = -\theta^{21} \equiv \theta$, $\theta^{\mu\nu} = 0$ otherwise. We find

$$\begin{aligned}\langle 0|\Delta|pp'\rangle &= \delta(x^0-y^0)(\mathrm{e}^{-\mathrm{i}p\cdot x-\mathrm{i}p'\cdot y} + \mathrm{e}^{-\mathrm{i}p'\cdot x-\mathrm{i}p\cdot y}) \\ &\quad\times \frac{\mathrm{i}}{(2\pi)^{2d-1}}\int \mathrm{d}^{d-1}k\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}\cos\left(\frac{1}{2}\theta^{ij}k_ip_j\right)\cos\left(\frac{1}{2}\theta^{ij}k_ip'_j\right). \\ &= (\mathrm{e}^{-\mathrm{i}p\cdot x-\mathrm{i}p'\cdot y} + \mathrm{e}^{-\mathrm{i}p'\cdot x-\mathrm{i}p\cdot y})\frac{\mathrm{i}}{(2\pi)^7}\delta(x^0-y^0) \\ &\quad\times \sum_{s=\pm1,t=\pm1}\delta(x^1-y^1-s\theta(p_2+tp'_2)) \\ &\quad\times \delta(x^2-y^2-s\theta(p_1+tp'_1))\delta(x^3-y^3). \end{aligned} \tag{41}$$

It may appear that the causality violation term Δ is non-vanishing only for a specific combinations of coordinate differences and momenta. However, if we use wave packets for the external lines, there will be a region in the $x_1 - x_2$ plane for which Δ will be nonzero.

Example 2. Let us turn to the case of the noncommutative Yukawa theory.

$$\begin{aligned} S &= S_0 + S_I \\ S_0 &= \int \mathrm{d}^4x\left[\bar{\psi}[\mathrm{i}\partial\!\!\!/ - m]\psi + \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right]\right] \\ S_I &= \lambda\int \mathrm{d}^4x\,\bar{\psi}*\psi*\phi \equiv \lambda\int \mathrm{d}^4x\,L_I^*(x). \end{aligned}$$

We note that unlike the ϕ^4 theory, the interaction Lagrangian does contain coordinate ψ and momentum $\bar{\psi}$ at the same time[7]. On account of this, the commutator,

$$[L_I(x), L_I(y)]|_{x_0=y_0}$$

has terms containing a Dirac delta function $\delta^3(\mathbf{x}-\mathbf{y})$ and is zero when $x \neq y$. In a NCQFT, this delta function will get smeared and can be nonzero when $x_0 = y_0$, $x_3 = y_3$, $x_\perp \neq y_\perp$.

To emphasize the point, we note:

$$L_I^*(x_0,\mathbf{x})L_I^*(x_0,\mathbf{y}) \neq L_I^*(x_0,\mathbf{y})L_I^*(x_0,\mathbf{x})\mathbf{x} \neq \mathbf{y}. \tag{42}$$

[7] However, we note that the integral for S_I has no operator ordering problems.

We now compute the causality violation amplitude of (37) by taking the limit $x_0 \to y_0^+$. We have[8]

$$\begin{aligned}\Delta_1 &\equiv \tilde{T}[\bar{\psi}(x) * \psi(x) * \varphi(x)\bar{\psi}(y) * \psi(y) * \varphi(y)]\\ &\quad - T[\bar{\psi}(x) * \psi(x) * \varphi(x)\bar{\psi}(y) * \psi(y) * \varphi(y)]\\ &= \tfrac{1}{2}\hat{D}_1\hat{D}_2\{[\bar{\psi}_\alpha(y)\psi_\alpha(y_1), \bar{\psi}_\beta(x)\psi_\beta(x_1)]\varphi(x_2)\varphi(y_2)\\ &\quad + \text{a term involving}[\phi(x_2), \phi(y_2)]\}.\end{aligned}$$

With,

$$\begin{aligned}\hat{D}_1 &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}[\partial_\mu^x\partial_\nu^{x_1} + \partial_\mu^{x_1}\partial_\nu^{x_2} + \partial_\mu^x\partial_\nu^{x_2}]}\\ \hat{D}_2 &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}[\partial_\mu^y\partial_\nu^{y_1} + \partial_\mu^{y_1}\partial_\nu^{y_2} + \partial_\mu^y\partial_\nu^{y_2}]}.\end{aligned} \tag{43}$$

The term involving $[\phi(x_2), \phi(y_2)]$, (which is a c number) does not contribute to the following matrix element which we are about to calculate. We now calculate the matrix element of Δ_1 between a state containing two scalars with momenta l, l' and a fermion of momentum p and a state with only a fermion of momentum p'.

$$\begin{aligned}2\Delta_2 &\equiv 2\langle p', s|\Delta_1|p, s; l, l'\rangle\\ &= \hat{D}_1\hat{D}_2\{\langle p', s|[\bar{\psi}_\alpha(y)\psi_\alpha(y_1), \bar{\psi}_\beta(x)\psi_\beta(x_1)]|p, s\rangle\langle 0|\varphi(x_2)\varphi(y_2)|l, l'\rangle\}\\ &= \mathrm{e}^{-\mathrm{i}lx - \mathrm{i}l'y}\\ &\quad\times\begin{bmatrix}\mathrm{e}^{\mathrm{i}(p'\wedge l' - p\wedge l)} \times \mathrm{e}^{\mathrm{i}p'y - \mathrm{i}px}\prod_i \delta((x-y)_i - \theta^{ij}p_j + \theta^{ij}p'_j - \theta^{ij}l_j + \theta^{ij}l'_j)\\ -\mathrm{e}^{\mathrm{i}(p'\wedge l - p\wedge l')} \times \mathrm{e}^{\mathrm{i}p'x - \mathrm{i}py}\prod_i \delta((x-y)_i + \theta^{ij}p_j - \theta^{ij}p'_j - \theta^{ij}l_j + \theta^{ij}l'_j)\end{bmatrix}\\ &\quad\times \bar{u}^\alpha(p')\gamma^o_{\alpha\beta}u^\beta(p)\\ &\quad + \mathrm{e}^{-\mathrm{i}l'x - \mathrm{i}ly}\\ &\quad\times\begin{bmatrix}\mathrm{e}^{\mathrm{i}(p'\wedge l - p\wedge l')} \times \mathrm{e}^{\mathrm{i}p'y - \mathrm{i}px}\prod_i \delta((x-y)_i - \theta^{ij}p_j + \theta^{ij}p'_j + \theta^{ij}l_j - \theta^{ij}l'_j)\\ -\mathrm{e}^{\mathrm{i}(p'\wedge l' - p\wedge l)} \times \mathrm{e}^{\mathrm{i}p'x - \mathrm{i}py}\prod_i \delta((x-y)_i + \theta^{ij}p_j - \theta^{ij}p'_j + \theta^{ij}l_j - \theta^{ij}l'_j)\end{bmatrix}\\ &\quad\times \bar{u}^\alpha(p')\gamma^o_{\alpha\beta}u^\beta(p).\end{aligned}$$

This nonzero result implies that noncommutative Yukawa theory (which is a nonlocal theory in the sense of nonlocality via the interaction term), is causality violating in the case of the space–space noncommutativity (as well as for the spacetime noncommutativity). In this case too, if we consider the matrix elements between the wave packets states, rather than plane wave states, we will find a region of causality violation spread over a finite extent in the $x_1 - x_2$ plane.

We note that in both the cases in this section, the causality violation is seen to occur via a calculation in the lowest order. Hence, the results do not involve UV/IR mixing as such.

6. Measurement and the causality condition

We would like to formulate the condition under which two 'local' observables in a NCQFT, $O_1^*(x)$ and $O_2^*(y)$ are compatible, i.e. their measurements do not interfere with each other. We shall show that this information is already present in the causality condition (14). We shall first consider the possibility when both $O_1^* = O_2^* = O^* = -\mathrm{i}S_1$. Now, the perturbative U matrix differs from identity, $\mathcal{I}$, by the effect of interactions: i.e. it 'measures', if indirectly, the impact of an interaction perturbatively. (This is in the same sense that charge density is measured by

[8] In the present context, the path-integral method does not produce a T^* product in the conventional sense, but is nonetheless symmetric in x and y. Hence, we have changed the notation from $T^* \to \tilde{T}$.

perturbing the electrostatic potential, or $\theta_{\mu\nu}$ is measured by perturbing the gravitational field $h_{\mu\nu}$.) Thus, $\frac{\delta U}{\delta g(x)}$ measures the effect of observation of O at x. $\frac{\delta^2 U}{\delta g(x)\delta g(y)}$ with $x_0 < y_0$ has in it the information of measurement of $O^*(x)$ followed by $O^*(y)$, in the nature of a change in U. The effect of measurement of $O(x)$ alone (i.e. one interaction at x taking place), on U is to take U from[9]:

$$\mathcal{I} \to U = \left(\mathcal{I} + \frac{\delta U}{\delta g(x)}\delta g(x)\right) \tag{44}$$

and the effect of measurement of $O^*(y)$ alone is

$$U = \left(\mathcal{I} + \frac{\delta U}{\delta g(y)}\delta g(y)\right). \tag{45}$$

The two measurements are compatible if these two 'add up' to the net effect of the two successive measurements. In other words, the second-order terms, $(O\,[\delta g(x)\delta g(y)])$, in U agree with the compounded effect of two successive measurements:

$$\begin{aligned} &\text{second-order term in} \left(\mathcal{I} + \frac{\delta U}{\delta g(y)}\delta g(y)\right)\left(\mathcal{I} + \frac{\delta U}{\delta g(x)}\delta g(x)\right) \\ &\quad = \text{second-order term in} \left(\mathcal{I} + \frac{\delta U}{\delta g(x)}\delta g(x) + \frac{\delta U}{\delta g(y)}\delta g(y)\right. \\ &\quad\quad \left. + \frac{\delta^2 U}{\delta g(x)\delta g(y)}\delta g(x)\delta g(y)\right). \end{aligned} \tag{46}$$

This can be seen to be just the causality condition (14), expanded to $O(g^2)$ (recalling $U_1^\dagger = -U_1$). Thus, two observables $O^*(x)$ and $O^*(y)$ are compatible only if

$$\Delta \equiv T^*[O^*(x)O^*(y)] - T[O^*(x)O^*(y)] = 0. \tag{47}$$

This can easily be generalized to the case when arbitrary 'local' observables $O_1^*(x)$ and $O_2^*(y)$ are measured. We can introduce sources for $O_1^*(x)$ and $O_2^*(y)$ in the action:

$$S_J = S + \int \mathrm{d}^4x[J_1(x)O_1^*(x) + J_2(x)O_2^*(x)]. \tag{48}$$

We can now repeat the above argument, but apply it to $O\,[\delta J_1(x)\delta J_2(y)]$ terms. Thus, to summarize, 'local' observables $O_1^*(x)$ and $O_2^*(y)$ for a NCQFT are compatible only if,

$$\Delta \equiv T^*[O_1^*(x)O_2^*(y)] - T[O_1^*(x)O_2^*(y)] = 0. \tag{49}$$

7. Conclusions

In this section, we shall summarize our main results.

(1) We approached the question of causality violation in the noncommutative field theories with the space–space noncommutativity in an *ab initio* manner. We started from the Bogoliubov–Shirkov approach which has been formulated for commutative spacetime; made an appropriate generalization to the space–space noncommutativity. We then developed a criterion that characterizes causality violation applicable to such a field theory on such a noncommutative space. Unlike earlier criteria, our attempt is from first principles and uses the primary meaning of causality.

[9] Here, $\delta g(x)$ is concentrated around x and we are suppressing an integration.

(2) We applied the criterion to two simple examples. Both the examples contain operators that have a field and a conjugate momentum. Our calculation exhibits presence of causality violation in field theories with space–space noncommutativity.

(3) We elaborated the meaning of the criterion in terms of the compatibility of observables in a measurement process. Physically, when the causality criterion fails, measurement of $O_1(x)$ can produce non-trivial effects on a subsequent measurement of $O_2(y)$.

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[14] For this procedure of introducing a single θ-function, see e.g., Rim C, Seo Y and Hyung Yee J 2004 *Phys. Rev.* D **70** 025006